Fourier Analysis

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Review

Thm (Fourier inversion formula)

Let $f \in \mathcal{M}(\mathbb{R})$. Suppose that $\hat{f} \in \mathcal{M}(\mathbb{R})$.

Then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi, \quad \forall x \in \mathbb{R}.$$

$$\left(f = \check{\hat{f}} \right)$$

To prove the above theorem, let us introduce the following.

Def. Let $f, g \in \mathcal{M}(\mathbb{R})$. Set

$$f * g(x) := \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy.$$

Prop 2: Let $f, g \in \mathcal{M}(\mathbb{R})$. Then

(1) $f * g = g * f$.

(2) $f * g \in \mathcal{M}(\mathbb{R})$.

(3) $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$.

Pf: ① Fix $x \in \mathbb{R}$.

$$f*g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

$$\underset{\text{Letting } z=x-y}{=} \int_{+\infty}^{-\infty} f(z)\, g(x-z) \cdot (-1)\, dz$$

$$= \int_{-\infty}^{\infty} g(x-z)\, f(z)\, dz$$

$$= g*f(x).$$

②: $f*g \in M(\mathbb{R})$.

$$f*g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$$

$$= \int_{|y| \le \frac{|x|}{2}} + \int_{|y| > \frac{|x|}{2}} f(x-y)\, g(y)\, dy$$

$$= (\mathrm{I}) + (\mathrm{II}),$$

where $|(\mathrm{I})| \le \int_{|y| \le \frac{|x|}{2}} |f(x-y)|\, |g(y)|\, dy$

(Noticing $|x-y| \ge \frac{|x|}{2}$)

$$\leq \int_{|y| \leq \frac{|x|}{2}} \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \cdot |g(y)| \, dy$$

$$\left(|f(x-y)| \leq \frac{A}{1+|x-y|^2} \right)$$

$$\leq \frac{A}{1+\frac{x^2}{4}} \cdot \int_{\mathbb{R}} |g(y)| \, dy$$

$$\leq \frac{\widetilde{A}}{1+x^2}$$

$$|(\mathrm{II})| \leq \int_{|y| > \frac{|x|}{2}} |f(x-y)| \, |g(y)| \, dy$$

$$\leq \int_{|y| > \frac{|x|}{2}} |f(x-y)| \cdot \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \, dy$$

$$\leq \frac{A}{1+\frac{x^2}{4}} \int_{-\infty}^{\infty} |f(x-y)| \, dy$$

$$= \frac{A}{1+\frac{x^2}{4}} \int_{-\infty}^{\infty} |f(y)| \, dy$$

$$\leq \frac{\tilde{\hat{A}}}{1+x^2}$$

Hence

$$|f * g(x)| \leq \frac{\tilde{\tilde{\hat{A}}}}{1+x^2}$$

We still need to show that

$f * g$ is cts on $\mathbb{R}$.

Notice that $f \in \mathcal{M}(\mathbb{R})$, so f is cts on $\mathbb{R}$

and $\lim\limits_{|x| \to \infty} f(x) = 0$.

This implies that f is unif cts on $\mathbb{R}$.

So $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon \quad \text{if} \quad |x - x'| < \delta.$$

Now for $x, x' \in \mathbb{R}$ with $|x-x'|<\delta$,

$$|f*g(x) - f*g(x')|$$

$$= \left| \int_{-\infty}^{\infty} (f(x-y) - f(x'-y))\, g(y)\, dy \right|$$

$$\leq \int_{-\infty}^{\infty} |f(x-y) - f(x'-y)|\, |g(y)|\, dy$$

$$\leq \varepsilon \int_{-\infty}^{\infty} |g(y)|\, dy.$$

This proves the unif continuity of $f*g$.

So $f*g \in \mathcal{M}(\mathbb{R})$.

③ : $\widehat{f*g}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$.

$$\widehat{f*g}(\xi) = \int_{-\infty}^{\infty} f*g(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \right) \cdot e^{-2\pi i \xi x}\, dx$$

(Recall <u>Fubini Thm</u> : Let $F(x,y) \in C(\mathbb{R}^2)$

suppose one of the following 3 holds

① $\iint_{\mathbb{R}^2} |F(x,y)|\, dx\, dy < \infty$;

② $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,y)|\, dy \right) dx < \infty$;

③ $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,y)|\, dx \right) dy < \infty$;

Then

$$\iint_{\mathbb{R}^2} F(x,y)\,dx\,dy$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y)\,dy \right) dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y)\,dx \right) dy.$$

)

By Fubini

$$\widehat{f*g}(\xi) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy \right) \cdot e^{-2\pi i \xi x}\,dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)\,g(y)\,e^{-2\pi i \xi x}\,dx \right) dy$$

(checking

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x-y)|\,|g(y)|\,dx\right)dy < \infty)$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)\,e^{-2\pi i\xi x}\,dx\right) g(y)\,dy$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)\cdot e^{2\pi i\xi(-y)}\,g(y)\,dy$$

$$= \hat{f}(\xi)\cdot\hat{g}(\xi).$$

$\blacksquare$

Def. (Good Kernel on $\mathbb{R}$)

A family of $(K_t)_{t\in(a,b)} \subset M(\mathbb{R})$ is said to be a good kernel on $\mathbb{R}$, as $t\to t_0$, if

(1) $$\int_{\mathbb{R}} K_t(x)\,dx = 1 \quad \text{for all } t\in(a,b).$$

(2) $$\int_{\mathbb{R}} |K_t(x)|\,dx \le M \quad \text{for all } t\in(a,b),$$

where $M>0$ is a constant.

(3) $\forall \delta>0$,

$$\int_{|x|>\delta} |K_t(x)|\,dx \to 0 \quad \text{as } t\to t_0.$$

Thm (Convergence thm about good kernels).

Let $(K_t)_{t\in(a,b)}$ be a good kernel on $\mathbb{R}$.

Let $f\in M(\mathbb{R})$. Then

$$K_t * f(x) \rightrightarrows f(x) \text{ on } \mathbb{R} \quad \text{as } t\to t_0$$

Thm (Multiplicative formula).

Let $f, g \in \mathcal{M}(\mathbb{R})$.

Then $$\int_{\mathbb{R}} f(x) \cdot \hat{g}(x)\, dx = \int_{\mathbb{R}} \hat{f}(x)\, g(x)\, dx.$$

Pf.

$$\int_{\mathbb{R}} f(x)\, \hat{g}(x)\, dx$$

$$= \int_{\mathbb{R}} f(x) \left(\int_{\mathbb{R}} g(y)\, e^{-2\pi i y x}\, dy \right) dx$$

(Noticing that $\int_{\mathbb{R}^2} |f(x)| \cdot |g(y)|\, dx\, dy < \infty$)

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)\, g(y)\, e^{-2\pi i y x}\, dx \right) dy$$

$$= \int_{\mathbb{R}} g(y) \cdot \hat{f}(y)\, dy. \quad \blacksquare$$